Equations in Cosmology

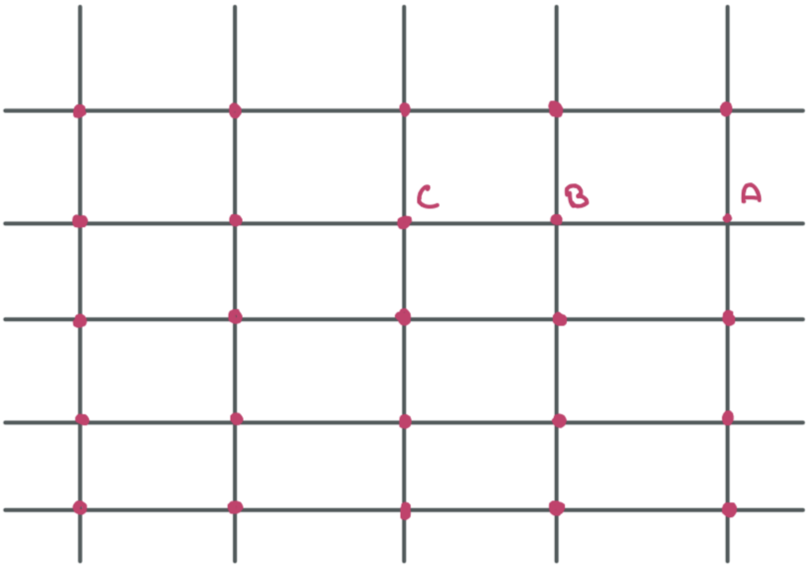
In this post, we will look at some of the equations in cosmology that determine the dynamics of the Universe.

1. Friedmann equation
2. Fluid equation
3. Acceleration equation
4. Equation of State

Friedmann equation

We know the Universe is expanding and that the rate at which two galaxies move apart is proportional to the distance between them.

Let's consider galaxies in the following grid



As the Universe expands the distance between A and B, lets call it D_{ab} increases, since D_{ac} is twice that of D_{ab}, it separation increases twice as fast.

$$D = a(t)\Delta x$$
$$D_{ab} = a(t)\Delta x_{ab}$$
$$D_{cb} = a(t)\Delta x_{cb}$$

$a(t)$ is the scale factor which is a function of time. Δx is the grid unit length. Let's now look at the velocity which is simply the time derivative of the distance.

$$D = a(t)\Delta x \text{ and } V = \dot{a}(t)\Delta x$$

Taking the ratio

$$\frac{V}{D} = \frac{\dot{a}(t)}{a(t)}$$

This gives

$V = H * D$ where $H = \frac{\dot{a}(t)}{a(t)}$ is the Hubble constant.

Consider the law of conservation of Energy

$$T + U = E$$

$$\frac{mv^2}{2} - \frac{GMm}{D} = E$$

here the first term is the kinetic energy, the second term is the potential energy and E represents the Total energy.

But we know $v = \dot{a}R$ and $D = a(t)R$, where R is just $\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$

$$\frac{\dot{a}^2(t)R^2}{1} - \frac{2GM}{a(t)R} = \frac{2E}{m},$$

$$\frac{\dot{a}^2(t)R^2}{1} - \frac{8\pi G a^2(t) R^2 \rho}{3} = \frac{2E}{m}$$

since $M = \frac{4\pi a^3(t) R^3 \rho}{3}$

Dividing the whole equation with $a^2(t)R^2$

$$\frac{\dot{a}^2(t)}{a^2(t)} - \frac{8\pi G\rho}{3} = \frac{2E}{ma^2(t)R^2}$$

Finally we arrive at the equation

$$\frac{\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G\rho}{3} - \frac{\kappa}{a^2(t)}$$

Where $\kappa = -\frac{2E}{mR^2}$ represents the curvature of the Universe, it is 1,0 and -1 for a positive, flat and negative curvature respectively. The current estimates predict the our Universe is most likely Flat.

> 💡 This equation is known as the Friedmann equation. Lets now look at the Fluid equation.

Fluid equation

Consider the first law of thermodynamics

$$dE + PdV = TdS$$

For a reversible system, the change in entropy $dS = 0$

$$dE = -PdV$$

$$E = mc^2 = \frac{4\pi a^3(t)R^3\rho c^2}{3}$$

Since $m = \rho * V$ where ρ is the Energy density and

$$V = \frac{4\pi a^3(t)R^3}{3}$$

Ignoring R^3 since it will get cancelled from both sides

$$\frac{dE}{dt} = \dot{E} = \frac{4\pi\rho c^2 a^2(t)\dot{a}}{} + \frac{4\pi a^3(t)c^2\dot{\rho}}{3}$$

And

$$\frac{dV}{dt} = \dot{V} = 4\pi a^2(t)\dot{a}(t)$$

We can now write

$$4\pi\rho c^2 a^2(t)\dot{a} + \frac{4\pi a^3(t)c^2\dot{\rho}}{3} + 4P\pi a^2(t)\dot{a}(t) = 0$$

Cancelling the terms

$$\rho\dot{a} + \frac{a\dot{\rho}}{3} + \frac{P\dot{a}(t)}{c^2} = 0$$

$$\dot{\rho} + 3\frac{\dot{a}(t)}{a}[\rho + \frac{P}{c^2}] = 0$$

💡 This equation is known as the Fluid equation. Finally let's derive the acceleration equation.

Acceleration equation.

Given the Friedmann equation and the Fluid equation, we can now derive the acceleration equation.

I can write the Friedmann equation like thiis

> $$\dot{a}^2(t) = \frac{8\pi G\rho a^2(t)}{3} - \kappa$$

Taking time derivative both sides

> $$2\dot{a}(t)\ddot{a}(t) = \frac{16\pi G\rho a(t)\dot{a}(t)}{3} + \frac{8\pi G a^2(t)\dot{\rho}}{3}$$

Substituting the value of $\dot{\rho}$ from the Fluid equation.

$$\ddot{a}(t) = \frac{8\pi G\rho a(t)}{3} - 4\pi G a(t)[\rho + \frac{P}{c^2}]$$

Dividing by the scale factor $a(t)$ both sides, putting speed of light c = 1 and simplifying, we arrive at the acceleration equation.

> $$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}[\rho + 3P]$$

💡 This equation is known as the acceleration equation.

Lets write all the three equations together, the Friedmann equation, the Fluid equation and the acceleration equation.

$$\frac{\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G\rho}{3} - \frac{\kappa}{a^2(t)}$$

$$\dot{\rho} + 3\frac{\dot{a}(t)}{a}[\rho + P] = 0$$

$$\frac{\ddot{a}(t)}{a(t)} + \frac{4\pi G}{3}[\rho + 3P]$$

So now we have three equations and only two of them are independent. We just derived the acceleration equation using the fluid and the friedmann equation. Since we have three unknows $a(t)$, $\rho(t)$ and P, we would need one more equation for consistency. This fourth equation, also known as equation of state is a relation between the energy density $\rho(t)$ and the pressure P.

Equation of state

$$P - w\rho = 0$$

Here w is a dimensionless quantity that depends on the stuff that fills the Universe. For a Universe dominated by radiation w will be different compared to for a Universe which is matter dominated.

These four equations together determine the dynamics of the Universe.

References

1. Introduction to cosmology by Barbara Ryden